

# Geometric RSK correspondence, Whittaker functions and symmetrized random polymers

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**ABSTRACT.** We show that the geometric lifting of the RSK correspondence introduced by A.N. Kirillov (2001) is volume preserving with respect to a natural product measure on its domain, and that the integrand in Givental’s integral formula for  $GL(n, \mathbb{R})$ -Whittaker functions arises naturally in this context. This yields a new proof (and generalisation) of Stade’s Whittaker integral identity, which can be seen as the analogue of the Cauchy-Littlewood identity in this setting. We also consider the restriction of the geometric RSK mapping to symmetric matrices and show that the volume preserving property continues to hold. The corresponding Whittaker integral identity involves only a single Whittaker function. As an application, we determine the law of the partition function for a random directed polymer model with log-gamma weights which are constrained to be symmetric about the main diagonal, with an additional ‘pinning’ factor on the main diagonal.

## 1. Introduction

The Robinson-Schensted-Knuth (RSK) correspondence is a combinatorial mapping which plays a fundamental role in the theory of Young tableaux, symmetric functions and representation theory. It is deeply connected with Schur functions and provides a combinatorial framework for understanding the Cauchy-Littlewood identity and Schur measures on integer partitions. It is also the basic structure which lies behind the solvability of a particular family of combinatorial models in probability and statistical physics including longest increasing subsequence problems, directed last passage percolation in 1+1 dimensions and the totally asymmetric simple exclusion process, see for example [1, 3, 16, 21].

The RSK correspondence can be defined by expressions in the max-plus semi-ring. Replacing these expressions by their analogues in the usual algebra, A.N. Kirillov [18] introduced a geometric lifting of the RSK correspondence which he called the ‘tropical RSK correspondence’. Unfortunately, for many readers nowadays, the word ‘tropical’ indicates just the opposite, so to avoid confusion we will refer to Kirillov’s construction as the *geometric*

RSK (gRSK) correspondence, as in the theory of *geometric crystals* [6, 7], which is very closely related.

The geometric RSK correspondence is a bijective mapping from  $(\mathbb{R}_{>0})^{n \times m}$  onto itself. It was introduced by Kirillov [18] for square matrices ( $n = m$ ) and generalized to the rectangular setting by Noumi and Yamada [19]. In the paper [11] it was shown that there is a fundamental connection between the gRSK correspondence and  $GL(n, \mathbb{R})$ -Whittaker functions, analogous to the well-known connection between the RSK correspondence and Schur functions. In particular, it is argued there that the analogue of the Cauchy-Littlewood identity in the setting of gRSK can be seen as a generalisation of a Whittaker integral identity which was originally conjectured by Bump [10] and later proved by Stade [24]. The connection to Whittaker functions gives rise to a natural family of measures (Whittaker measures) which play a similar role in this setting to Schur measures on integer partitions. It also has applications to random polymers. In the paper [11], an explicit integral formula is obtained for the Laplace transform of the law of the partition function associated with a random directed polymer model on the two-dimensional lattice with log-gamma weights introduced in [23]. For related recent developments, see [20, 8, 9].

In the present work, we first clarify the results of [11] by showing:

- (a) the gRSK mapping is volume preserving with respect to the product measure  $\prod_{ij} dx_{ij}/x_{ij}$  on  $(\mathbb{R}_{>0})^{n \times m}$ , and
- (b) the integrand in Givental's integral formula for  $GL(n, \mathbb{R})$ -Whittaker functions [14, 15] arises naturally through the application of the gRSK mapping (see Theorem 3.2 below).

The volume preserving property can be seen as a consequence of (a re-formulation of) the 'row-insertion' procedure introduced by Noumi and Yamada in [19] to define the gRSK mapping. Our results also give rise to a new proof of Stade's identity, with some restrictions on the parameters.

The second aim of this paper is to initiate a program of understanding the gRSK mapping in the presence of symmetry constraints in much the same spirit as the work of Baik and Rains [4, 5, 2] on longest increasing subsequence and last passage percolation problems. Here we consider one particular symmetry, namely the restriction of gRSK to symmetric matrices. We show that the volume preserving property continues to hold in this setting and deduce the analogue of the Whittaker measure. The corresponding Whittaker integral identity (Corollary 5.5) involves only a single Whittaker function. As an application we determine the law of the partition function for a random directed polymer model with log-gamma weights which are constrained to be symmetric about the main diagonal, with an additional 'pinning' factor on the main diagonal.

The outline of the paper is as follows. In the next section we give some background on Whittaker functions, introduce a generalisation of these functions and explain how these functions can be regarded as generating functions for *patterns*. This interpretation can be seen as a generalisation of Givental's integral formula [14] and is analogous to the combinatorial interpretation of Schur functions as generating functions for semistandard Young tableaux. In Section 3 we introduce a re-formulation of Noumi and Yamada's dynamical description the gRSK correspondence and use this to establish several basic results. In particular, we show that the gRSK mapping is volume-preserving with respect to a natural product measure on  $(\mathbb{R}_{>0})^{n \times m}$  and establish a fundamental identity (Theorem 3.2) which explains the appearance of Whittaker functions in this setting. This clarifies and extends earlier results from [11] and yields a new proof of Stade's Whittaker integral identity. In Section 4 we explain the relationship between our new description of gRSK and the row-insertion algorithm of Noumi and Yamada [19]. In Section 5, we consider the restriction of gRSK to symmetric matrices. We show that the volume preserving property continues to hold in this setting and deduce several new results, including a Whittaker integral identity (Corollary 5.5) which involves only a single Whittaker function.

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## 2. Whittaker functions and patterns

We begin by defining the following Baxter  $Q$ -type operators, as in [12, 13]. For  $\lambda \in \mathbb{C}$ ,  $x, y \in (\mathbb{R}_{>0})^n$ , define

$$Q_\lambda^n(x, y) = \left( \prod_{i=1}^n \frac{y_i}{x_i} \right)^\lambda \exp \left( - \sum_{i=1}^n \frac{y_i}{x_i} - \sum_{i=1}^{n-1} \frac{x_{i+1}}{y_i} \right).$$

For  $\lambda \in \mathbb{C}$ ,  $x \in (\mathbb{R}_{>0})^n$  and  $y \in (\mathbb{R}_{>0})^{n-1}$ , define

$$Q_\lambda^{n,n-1}(x, y) = \left( \frac{\prod_{i=1}^{n-1} y_i}{\prod_{i=1}^n x_i} \right)^\lambda \exp \left( - \sum_{i=1}^{n-1} \frac{y_i}{x_i} - \sum_{i=1}^{n-1} \frac{x_{i+1}}{y_i} \right).$$

We regard these as integral operators: for suitable test functions,

$$Q_\lambda^n f(x) = \int_{(\mathbb{R}_{>0})^n} Q_\lambda^n(x, y) f(y) \prod_{i=1}^n \frac{dy_i}{y_i},$$

$$Q_\lambda^{n,n-1} f(x) = \int_{(\mathbb{R}_{>0})^{n-1}} Q_\lambda^{n,n-1}(x, y) f(y) \prod_{i=1}^{n-1} \frac{dy_i}{y_i}.$$

Define  $\Psi_\lambda^n(x)$ ,  $\lambda \in \mathbb{C}^n$ ,  $x \in (\mathbb{R}_{>0})^n$  recursively by setting  $\Psi_\lambda^1(x) = x^{-\lambda}$  and, for  $n \geq 2$ ,

$$(2.1) \quad \Psi_{\lambda_1, \dots, \lambda_n}^n = Q_{\lambda_n}^{n, n-1} \Psi_{\lambda_1, \dots, \lambda_{n-1}}^{n-1}.$$

The functions  $\Psi_\lambda^n$  are  $GL(n, \mathbb{R})$ -Whittaker functions. The above definition is essentially Givental's integral formula for these functions [14, 15] (see also [12, 13]).

We will also consider the following generalisation of these functions. For  $\lambda \in \mathbb{C}^n$ ,  $x \in (\mathbb{R}_{>0})^n$  and  $s \in \mathbb{C}$ , define

$$(2.2) \quad \Psi_{\lambda;s}^n(x) = e^{-s/x_n} \Psi_{\lambda}^n(x);$$

for  $\lambda \in \mathbb{C}^{n+k}$ ,  $k \geq 1$ , and  $\Re s > 0$ , define

$$(2.3) \quad \Psi_{\lambda;s}^n = Q_{\lambda_{n+k}}^n Q_{\lambda_{n+k-1}}^n \cdots Q_{\lambda_{n+1}}^n \Psi_{\lambda_1, \dots, \lambda_n; s}^n.$$

It is straightforward to see that  $\Psi_{\lambda;s}^n(x)$  is well-defined, as an absolutely convergent integral, for each  $x \in \mathbb{R}^n$ . The functions  $\Psi_{\lambda;s}^n$  can be regarded as generating functions for ‘patterns’, as we shall now explain.

Let  $x \in (\mathbb{R}_{>0})^n$ . We define a *pattern*  $P$  with *shape*  $\text{sh } P = x$  and *height*  $h \geq n$  to be an array of positive real numbers

$$P = \begin{pmatrix} z_{11} & & & \\ & z_{22} & & \\ & & \ddots & \\ & & & z_{nn} \end{pmatrix}$$

with bottom row  $z_{h\cdot} = x$ . The range of indices is

$$L(n, h) = \{(i, j) : 1 \leq i \leq h, 1 \leq j \leq i \wedge n\}.$$

If  $h = n$  then  $P$  is a *triangle* in the sense of Kirillov [18]. Fix a pattern  $P$  as above. Set  $\rho_0 = 1$  and, for  $1 \leq i \leq h$ ,  $\rho_i = \prod_{j=1}^{i \wedge n} z_{ij}$  and  $\tau_i = \rho_i / \rho_{i-1}$ . We shall refer to  $\tau$  as the *type* of  $P$  and write  $\tau = \text{type } P$ . For  $\alpha \in \mathbb{C}^h$  define

$$P^\alpha = \prod_{i=1}^h \tau_i^{\alpha_i}.$$

For  $s \in \mathbb{C}$ , define

$$(2.4) \quad \mathcal{F}_s(P) = \frac{s}{z_{nn}} + \sum_{(i,j) \in L(n,h)} \frac{z_{i-1,j} + z_{i+1,j+1}}{z_{ij}}$$

with the convention that  $z_{ij} = 0$  if  $(i, j) \notin L(n, h)$ . Denote by  $\Pi^h(x)$  the set of patterns with shape  $x$  and height  $h$ . Then, for  $\lambda \in \mathbb{C}^h$  and  $\Re s > 0$  (this

condition is only required if  $h > n$ )

$$(2.5) \quad \Psi_{\lambda;s}^n(x) = \int_{\Pi^h(x)} P^{-\lambda} e^{-\mathcal{F}_s(P)} dP$$

where

$$dP = \prod_{(i,j) \in L(n,h-1)} \frac{dz_{ij}}{z_{ij}}.$$

This formula is just a re-writing of the above definition (2.3) of  $\Psi_{\lambda;s}^n$ .

We remark that, although it is not obvious from the above definition, the function  $\Psi_{\lambda}^n$  is invariant under permutations of the indices  $\lambda_1, \dots, \lambda_n$  [17, 13]. In fact, the same is true of the function  $\Psi_{\lambda;s}^n$ , where  $\lambda \in \mathbb{C}^{n+k}$ ,  $k \geq 1$  and  $\Re s > 0$ . That is,  $\Psi_{\lambda;s}^n$  is invariant under permutations of the indices  $\lambda_1, \dots, \lambda_{n+k}$ . This follows from the definition (2.3), using the relation

$$(2.6) \quad Q_a^n R_s^n Q_b^n = Q_b^n R_s^n Q_a^n,$$

where  $R_s$  denotes multiplication by the function  $e^{-s/x_n}$ , and the invariance of  $\Psi_{\lambda_1, \dots, \lambda_n}^n$  under permutations of  $\lambda_1, \dots, \lambda_n$ . The relation (2.6) is a straightforward extension of the commutativity property  $Q_a^n Q_b^n = Q_b^n Q_a^n$  obtained in [13, Theorem 2.3].

There is a Plancherel theorem for the Whittaker functions [25, 22, 17], which states that the integral transform

$$\hat{f}(\lambda) = \int_{(\mathbb{R}_{>0})^n} f(x) \Psi_{\lambda}^n(x) \prod_{i=1}^n \frac{dx_i}{x_i}$$

defines an isometry from  $L_2((\mathbb{R}_{>0})^n, \prod_{i=1}^n dx_i/x_i)$  onto  $L_2^{sym}(\iota \mathbb{R}^n, s_n(\lambda) d\lambda)$ , where  $L_2^{sym}$  is the space of  $L_2$  functions which are symmetric in their variables,  $\iota = \sqrt{-1}$  and

$$s_n(\lambda) = \frac{1}{(2\pi\iota)^n n!} \prod_{i \neq j} \Gamma(\lambda_i - \lambda_j)^{-1}.$$

We note here, for later reference, that for  $a > 0$  we have

$$(2.7) \quad \Psi_{\alpha}^n(ax) = a^{-\sum_i \alpha_i} \Psi_{\alpha}^n(x),$$

and, if we set  $x'_i = 1/x_{n-i+1}$ , then

$$(2.8) \quad \Psi_{\lambda}^n(x) = \Psi_{-\lambda}^n(x').$$

These identities follow easily from the definitions.

### 3. Geometric RSK correspondence

The geometric RSK (gRSK) correspondence is a bijective mapping

$$T : (\mathbb{R}_{>0})^{n \times m} \rightarrow (\mathbb{R}_{>0})^{n \times m}.$$

It was introduced by Kirillov [18] as a geometric lifting of the RSK correspondence and further studied by Noumi and Yamada [19]. We will define

it here via a sequence of ‘local moves’ on matrix elements. This is essentially a reformulation of the row-insertion procedure introduced in [19], as will be explained in Section 4 below.

For each  $2 \leq i \leq n$  and  $2 \leq j \leq m$  define a mapping  $l_{ij}$  which takes as input a matrix  $X = (x_{ij}) \in (\mathbb{R}_{>0})^{n \times m}$  and replaces the submatrix

$$\begin{pmatrix} x_{i-1,j-1} & x_{i-1,j} \\ x_{i,j-1} & x_{ij} \end{pmatrix}$$

of  $X$  by its image under the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} bc/(ab+ac) & b \\ c & d(b+c) \end{pmatrix},$$

and leaves the other elements unchanged. For  $2 \leq i \leq n$  and  $2 \leq j \leq m$ , define  $l_{i1}$  to be the mapping that replaces the element  $x_{i1}$  by  $x_{i-1,1}x_{i1}$  and  $l_{1j}$  to be the mapping that replaces the element  $x_{1j}$  by  $x_{1,j-1}x_{1j}$ . For convenience define  $l_{11}$  to be the identity map. For  $1 \leq i \leq n$  and  $1 \leq j \leq m$ , set

$$\pi_i^j = l_{ij} \circ \cdots \circ l_{i1},$$

and, for  $1 \leq i \leq n$ ,

$$R_i = \begin{cases} \pi_1^{m-i+1} \circ \cdots \circ \pi_i^m & i \leq m \\ \pi_{i-m+1}^1 \circ \cdots \circ \pi_i^m & i \geq m. \end{cases}$$

The mapping  $T$  is defined by

$$(3.1) \quad T = R_n \circ \cdots \circ R_1.$$

For example, suppose  $n = m = 2$ . Then

$$R_1 = \pi_1^2 = l_{12} \circ l_{11} = l_{12}, \quad R_2 = \pi_1^1 \circ \pi_2^2 = l_{11} \circ l_{22} \circ l_{21} = l_{22} \circ l_{21}$$

and so

$$T = R_2 \circ R_1 = l_{22} \circ l_{21} \circ l_{12}.$$

Here is an illustration:

$$T : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{l_{12}} \begin{pmatrix} a & ab \\ c & d \end{pmatrix} \xrightarrow{l_{21}} \begin{pmatrix} a & ab \\ ac & d \end{pmatrix} \xrightarrow{l_{22}} \begin{pmatrix} bc/(b+c) & ab \\ ac & ad(b+c) \end{pmatrix}.$$

Each matrix  $X \in (\mathbb{R}_{>0})^{n \times m}$  can be identified with a pair of patterns  $(P, Q)$  with respective heights  $m$  and  $n$ , and common shape

$$\text{sh } P = \text{sh } Q = (x_{nm}, x_{n-1,m-1}, \dots, x_{n-p+1,m-p+1}),$$

where  $p = n \wedge m$ , as illustrated in the following example:

$$X = \begin{array}{ccc} & & x_{31} \\ & x_{21} & x_{32} \\ x_{11} & & x_{22} \\ & x_{12} & \end{array}$$

$$P = \begin{array}{ccc} & x_{31} & \\ x_{21} & & x_{32} \end{array}, \quad Q = \begin{array}{ccc} & x_{12} & \\ x_{11} & & x_{22} \\ & x_{21} & x_{32} \end{array}$$

$$\text{sh } P = \text{sh } Q = (x_{32}, x_{21}).$$

In the following, we will simply write  $X = (P, Q)$  to indicate that  $X$  is identified with the pair  $(P, Q)$ .

The mappings  $R_i$  defined above can also be written as

$$R_i = \rho_m^i \circ \cdots \circ \rho_2^i \circ \rho_1^i$$

where

$$\rho_j^i = \begin{cases} l_{1,j-i+1} \circ \cdots \circ l_{i-1,j-1} \circ l_{ij} & i \leq j \\ l_{i-j+1,1} \circ \cdots \circ l_{i-1,j-1} \circ l_{ij} & i \geq j. \end{cases}$$

Here we are just using the obvious fact that  $l_{ij} \circ l_{i'j'} = l_{i'j'} \circ l_{ij}$  whenever  $|i - i'| + |j - j'| > 2$ . This representation is closely related to the Bender-Knuth transformations, as we shall now explain. For each  $1 \leq i \leq n$  and  $1 \leq j \leq m$ , denote by  $b_{ij}$  the map on  $(\mathbb{R}_{>0})^{n \times m}$  which takes a matrix  $X = (x_{qr})$  and replaces the entry  $x_{ij}$  by

$$(3.2) \quad x'_{ij} = \frac{1}{x_{ij}}(x_{i,j-1} + x_{i-1,j}) \left( \frac{1}{x_{i+1,j}} + \frac{1}{x_{i,j+1}} \right)^{-1},$$

leaving the other entries unchanged, with the conventions that  $x_{0j} = x_{i0} = 0$ ,  $x_{n+1,j} = x_{i,m+1} = \infty$  for  $1 < i < n$  and  $1 < j < m$ , and  $x_{10} + x_{01} = x_{n+1,m}^{-1} + x_{n,m+1}^{-1} = 1$ . Denote by  $r_j$  the map which replaces the entry  $x_{nj}$  by  $x_{n,j+1}/x_{nj}$  if  $j < m$  and  $1/x_{nm}$  if  $j = m$ , leaving the other entries unchanged. For  $j \leq m$ , define

$$(3.3) \quad \tau_j = \begin{cases} b_{n-j+1,1} \circ \cdots \circ b_{n-1,j-1} \circ b_{nj} & j \leq n \\ b_{1,j-n+1} \circ \cdots \circ b_{n-1,j-1} \circ b_{nj} & j \geq n. \end{cases}$$

It is straightforward from the definitions to see that  $\rho_j^n = \tau_j \circ r_j$ . Now, observe that if  $X = (P, Q)$ , then for each  $j < m$ ,  $\tau_j(X) = (t_j(P), Q)$  where  $t_j$  is defined by this relation. It is easy to see that the mappings  $b_{ij}$ ,  $\tau_j$  and  $t_j$  are all involutions.

In the case  $n = m$ , the mappings  $t_1, \dots, t_{n-1}$  are the analogues of the Bender-Knuth transformations in this setting, as discussed in [18]. In this case, if we define, for  $i < n$ ,

$$(3.4) \quad q_i = t_1 \circ (t_2 \circ t_1) \circ \cdots \circ (t_i \circ \cdots \circ t_1),$$

then  $s_i = q_i \circ t_1 \circ q_i$ ,  $i < n$ , satisfy the same relations as the adjacent transpositions  $(i, i+1)$  in the symmetric group  $S_n$  and hence define an action of  $S_n$  on the set of triangles of height  $n$ . The mapping  $q_{n-1}$  is the analogue of Schützenberger's involution in this setting.

An immediate consequence of the above re-formulation of gRSK is the following volume preserving property. Denote the input matrix by  $W = (w_{ij}) \in (\mathbb{R}_{>0})^{n \times m}$  and the output matrix by  $T = T(W) = (t_{ij}) \in (\mathbb{R}_{>0})^{n \times m}$ .

**THEOREM 3.1.** *The gRSK mapping in logarithmic variables*

$$(\log w_{ij}, 1 \leq i \leq n, 1 \leq j \leq m) \mapsto (\log t_{ij}, 1 \leq i \leq n, 1 \leq j \leq m)$$

has Jacobian  $\pm 1$ .

**PROOF.** It is easy to see that the Jacobians of the mappings  $l_{ij}$  in logarithmic variables are all  $\pm 1$ . This follows from the fact that the mappings

$$(\log a, \log b) \mapsto (\log a, \log a + \log b)$$

$$(\log a, \log b, \log c, \log d) \mapsto (\log(bc/(ab + ac)), \log b, \log c, \log(db + dc))$$

each have Jacobian  $\pm 1$ . The result follows from the definition (3.1) of  $T$ .  $\square$

We remark that, by a similar argument it can be seen that the involutions  $q_i$ ,  $i < n$ , on the set of triangles of height  $n$ , all have Jacobian  $\pm 1$  in logarithmic variables.

We recall here some basic properties of the gRSK map  $T$ , which are either obvious from the definitions or proved in the papers [18, 19]. Suppose  $W \in (\mathbb{R}_{>0})^{n \times m}$  and  $T = T(W) = (P, Q)$ . If we define row and column products  $R_i = \prod_j w_{ij}$  and  $C_j = \prod_i w_{ij}$ , then  $\text{type } Q = R$  and  $\text{type } P = C$ . Note that this implies, for  $\lambda \in \mathbb{C}^m$  and  $\nu \in \mathbb{C}^n$ ,

$$(3.5) \quad \prod_{ij} w_{ij}^{-\nu_i - \lambda_j} = \prod_i R_i^{-\nu_i} \prod_j C_j^{-\lambda_j} = P^{-\lambda} Q^{-\nu}.$$

Also, the following symmetries hold:

$$\begin{aligned} T(W^t) &= T(W)^t; \\ T(W) = (P, Q) &\iff T(W^t) = (Q, P); \\ W \text{ is symmetric} &\iff T \text{ is symmetric} \iff P = Q; \\ W \text{ is symmetric across the anti-diagonal} &\iff Q = q_{n-1}(P). \end{aligned}$$

The connection to directed polymers is via the following formula for  $t_{nm}$ :

$$t_{nm} = \sum_{\pi \in \Pi_{n,m}} \prod_{(i,j) \in \pi} w_{ij},$$

where  $\Pi_{n,m}$  is the set of directed nearest-neighbor lattice paths in  $\mathbb{Z}^2$  from  $(1, 1)$  to  $(n, m)$ , that is, the set of paths  $\pi = \{\pi(1), \pi(2), \dots, \pi(n+m-1)\}$  such that  $\pi(1) = (1, 1)$ ,  $\pi(n+m-1) = (n, m)$  and  $\pi(k+1) - \pi(k) \in \{(1, 0), (0, 1)\}$  for  $1 \leq k < n+m-1$ . We shall refer to the variable  $t_{nm}$  as the *polymer partition function*. In fact, the remaining entries of  $T = (P, Q)$  can also be expressed in terms of similar partition functions, as follows. For  $1 \leq k \leq m$  and  $1 \leq r \leq n \wedge k$ ,

$$(3.6) \quad t_{n-r+1, k-r+1} \cdots t_{n-1, k-1} t_{nk} = \sum_{(\pi_1, \dots, \pi_r) \in \Pi_{n,k}^{(r)}} \prod_{(i,j) \in \pi_1 \cup \dots \cup \pi_r} w_{ij},$$



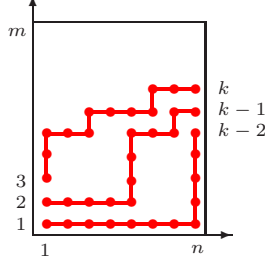


FIGURE 1. Three paths  $(\pi_1, \pi_2, \pi_3)$  of a particular 3-tuple in  $\Pi_{n,k}^{(3)}$  in an  $n \times m$  weight matrix. Note that the picture is in Cartesian coordinates. The paths start at the lower left at  $(1, 1)$ ,  $(1, 2)$  and  $(1, 3)$  and end at the upper right at  $(n, k-2)$ ,  $(n, k-1)$ ,  $(n, k)$ .

where  $\Pi_{n,k}^{(r)}$  denotes the set of  $r$ -tuples of non-intersecting directed nearest-neighbor lattice paths  $\pi_1, \dots, \pi_r$  starting at positions  $(1, 1), (1, 2), \dots, (1, r)$  and ending at positions  $(n, k-r+1), \dots, (n, k-1), (n, k)$ . (See Figure 1. When we use the path representation we draw the weight matrix in Cartesian coordinates.) This determines the entries of  $P$ . The entries of  $Q$  are given by similar formulae using  $T(W^t) = (Q, P)$ . We note here the following identity, which follows from the above lattice path representation for  $T$ : setting  $p = n \wedge m$ , we have

$$(3.7) \quad \sum_{i=1}^p \frac{1}{w_{i,p-i+1}} = \frac{1}{t_{11}}.$$

To see this if  $n \leq m$ , take the ratio of (3.6) for  $\Pi_{n,n}^{(n-1)}$  and  $\Pi_{n,n}^{(n)}$ . In the opposite case apply the same to  $W^t$ .

Now, for  $X \in (\mathbb{R}_{>0})^{n \times m}$  and  $s \in \mathbb{C}$ , define

$$(3.8) \quad \mathcal{E}_s(X) = \frac{s}{x_{11}} + \sum_{ij} \frac{x_{i-1,j} + x_{i,j-1}}{x_{ij}},$$

where the summation is over  $1 \leq i \leq n$ ,  $1 \leq j \leq m$  with the conventions  $x_{ij} = 0$  for  $i = 0$  or  $j = 0$ . Note that, if  $X = (P, Q)$ , then

$$\mathcal{E}_s(X) = \begin{cases} \mathcal{F}_0(P) + \mathcal{F}_s(Q) & n \geq m \\ \mathcal{F}_s(P) + \mathcal{F}_0(Q) & n \leq m \end{cases}$$

where  $\mathcal{F}_s$  is defined by (2.4). An important property of the maps  $b_{ij}$  defined by (3.2) above is that they preserve the quantity  $\mathcal{E}_0(X)$ , that is,  $\mathcal{E}_0 \circ b_{ij} = \mathcal{E}_0$ . To see this, recall that the map  $b_{ij}$  takes a matrix  $X = (x_{qr})$  and replaces the entry  $x_{ij}$  by

$$x'_{ij} = \frac{1}{x_{ij}} (x_{i,j-1} + x_{i-1,j}) \left( \frac{1}{x_{i+1,j}} + \frac{1}{x_{i,j+1}} \right)^{-1},$$

leaving the other entries unchanged, with the conventions that  $x_{0j} = x_{i0} = 0$ ,  $x_{n+1,j} = x_{i,m+1} = \infty$  for  $1 < i < n$  and  $1 < j < m$ , and  $x_{10} + x_{01} = x_{n+1,m}^{-1} + x_{n,m+1}^{-1} = 1$ . It is readily verified that

$$(3.9) \quad \frac{x'_{i,j-1} + x'_{i-1,j}}{x'_{ij}} + \frac{x'_{ij}}{x'_{i+1,j}} + \frac{x'_{ij}}{x'_{i,j+1}} = \frac{x_{i,j-1} + x_{i-1,j}}{x_{ij}} + \frac{x_{ij}}{x_{i+1,j}} + \frac{x_{ij}}{x_{i,j+1}}$$

with the conventions that  $x_{0j} = x_{i0} = x'_{0j} = x'_{i0} = 0$  and  $x_{n+1,j} = x_{i,m+1} = x'_{n+1,j} = x'_{i,m+1} = \infty$  for each  $i$  and  $j$ . This implies  $\mathcal{E}_0(b_{ij}(X)) = \mathcal{E}_0(X)$ . We remark that, in particular, this implies  $\mathcal{E}_0 \circ \tau_j = \mathcal{E}_0$ ,  $\mathcal{F}_0 \circ t_j = \mathcal{F}_0$  for all  $j < m$  and, in the case  $m = n$ ,  $\mathcal{F}_0 \circ q_{n-1} = \mathcal{F}_0$ , where  $q_{n-1}$  is the geometric analogue of Schützenberger's involution defined by (3.4).

The cornerstone of the present paper is the following identity which, combined with Theorem 3.1, explains the appearance of  $GL(n, \mathbb{R})$ -Whittaker functions in the context of geometric RSK.

**THEOREM 3.2.** *Let  $W \in (\mathbb{R}_{>0})^{n \times m}$ ,  $T = T(W)$  and  $s \in \mathbb{C}$ . Then*

$$\sum_{i=1}^p \frac{s}{w_{i,p-i+1}} + \sum' \frac{1}{w_{ij}} = \mathcal{E}_s(T),$$

where  $p = n \wedge m$  and  $\sum'$  denotes the sum over  $1 \leq i \leq n$ ,  $1 \leq j \leq m$  such that  $j \neq p - i + 1$ .

**PROOF.** From the identity (3.7), we can assume without loss of generality that  $s = 1$ . We will prove the theorem by induction on  $n$  and  $m$ . The statement is immediate in the case  $n = m = 1$ . Write  $R_i = R_i^{n,m}$ ,  $T = T^{n,m}$  and  $\mathcal{E}_s^{n,m}$  for the mappings defined above. Recall that  $T^{m,n}(W^t) = [T^{n,m}(W)]^t$ , for any values of  $m$  and  $n$ . It therefore suffices to show that the proposition holds for  $T^{n,m}$ , assuming that  $n \geq m$  and that the proposition holds for  $T^{n-1,m}$ .

Write  $W_{n-1,m} = (w_{ij}, 1 \leq i \leq n-1, 1 \leq j \leq m)$ ,  $S = T^{n-1,m}(W_{n-1,m})$  and  $T = T^{n,m}(W)$ . Then

$$T = R_n^{n,m} \begin{pmatrix} S \\ w_{n1} \dots w_{nm} \end{pmatrix},$$

and we are required to show that

$$\mathcal{E}_1^{n,m}(T) = \mathcal{E}_1^{n-1,m}(S) + \sum_{j=1}^m \frac{1}{w_{nj}}.$$

Now,

$$R_n^{n,m} = \rho_m^n \circ \dots \circ \rho_2^n \circ \rho_1^n$$

where

$$\rho_k^n = \tau_k \circ r_k = b_{n-k+1,1} \circ \dots \circ b_{nk} \circ r_k.$$

Set

$$T^{(0)} = \begin{pmatrix} S \\ w_{n1} \dots w_{nm} \end{pmatrix}$$

and, for  $k = 1, \dots, m$ ,

$$T^{(k)} = \rho_k^n \circ \dots \circ \rho_2^n \circ \rho_1^n(T^{(0)}).$$

For  $X \in (\mathbb{R}_{>0})^{n \times m}$  and  $0 \leq k \leq m$ , define

$$\mathcal{E}^{n,m;k}(X) = \frac{1}{x_{11}} + \sum_{ij}^{(k)} \frac{x_{i-1,j} + x_{i,j-1}}{x_{ij}} + \sum_{j=k+1}^m \frac{1}{x_{nj}},$$

where  $X = (x_{ij})$  and the first summation is over pairs of indices  $(i, j)$  such that either  $1 \leq i < n$  and  $1 \leq j \leq m$  or  $i = n$  and  $1 \leq j \leq k$ , with the conventions  $x_{ij} = 0$  for  $i = 0$  or  $j = 0$ . Note that

$$\mathcal{E}^{n,m;0}(T^{(0)}) = \mathcal{E}_1^{n-1,m}(T) + \sum_{j=1}^m \frac{1}{w_{nj}}, \quad \mathcal{E}^{n,m;m}(T^{(m)}) = \mathcal{E}_1^{n,m}(T).$$

We will show that

$$(3.10) \quad \mathcal{E}^{n,m;k} \circ \rho_k^n = \mathcal{E}^{n,m;k-1}$$

for each  $k = 1, \dots, m$ . Note that this implies

$$\mathcal{E}^{n,m;k}(T^{(k)}) = \mathcal{E}^{n,m;k-1}(T^{(k-1)})$$

for each  $k = 1, \dots, m$ , and the statement of the theorem follows.

Let  $X = (x_{ij}) \in (\mathbb{R}_{>0})^{n \times m}$  and write

$$X' = (x'_{ij}) = \rho_k^n(X) = \tau_k \circ r_k(X).$$

Note that  $x'_{ij} = x_{ij}$  for all  $(i, j)$  except  $(n - q + 1, k - q + 1)$ ,  $1 \leq q \leq k$ . Applying  $b_{nk} \circ r_k$  gives the relation

$$\frac{x'_{n,k-1} + x'_{n-1,k}}{x'_{nk}} = \frac{1}{x_{nk}}.$$

The next three relations follow from the invariance of  $\mathcal{E}_0$  under the  $b_{ij}$  mappings as discussed earlier, see (3.9). If  $(i, j) = (n - q + 1, k - q + 1)$  for some  $1 < q < k$ , then

$$\frac{x'_{i,j-1} + x'_{i-1,j}}{x'_{ij}} + \frac{x'_{ij}}{x'_{i+1,j}} + \frac{x'_{ij}}{x'_{i,j+1}} = \frac{x_{i,j-1} + x_{i-1,j}}{x_{ij}} + \frac{x_{ij}}{x_{i+1,j}} + \frac{x_{ij}}{x_{i,j+1}}.$$

If  $k < n$ , then

$$\frac{x'_{n-k,1}}{x'_{n-k+1,1}} + \frac{x'_{n-k+1,1}}{x'_{n-k+2,1}} + \frac{x'_{n-k+1,1}}{x'_{n-k+1,2}} = \frac{x_{n-k,1}}{x_{n-k+1,1}} + \frac{x_{n-k+1,1}}{x_{n-k+2,1}} + \frac{x_{n-k+1,1}}{x_{n-k+1,2}};$$

If  $k = n$  (this can only occur if  $m = n$ ), then

$$\frac{1}{x'_{11}} + \frac{x'_{11}}{x'_{21}} + \frac{x'_{11}}{x'_{12}} = \frac{1}{x_{11}} + \frac{x_{11}}{x_{21}} + \frac{x_{11}}{x_{12}}.$$

It follows that  $\mathcal{E}^{n,m;k}(X') = \mathcal{E}^{n,m;k-1}(X)$ , as required.  $\square$

The analogue of Theorem 3.2 in the setting of the usual RSK correspondence is the following. Suppose we apply the continuous version of the RSK correspondence (as defined in Section 3.3 of [18] for square matrices, but easily extends to rectangular matrices) to a matrix  $X \in \mathbb{R}^{n \times m}$ . Then the output is a pair of Gelfand-Tsetlin patterns (that is, the interlacing conditions are satisfied) if, and only if, the entries of  $X$  are all non-negative. The corresponding property also holds in the discrete setting.

Let  $s > 0$  and consider the measure on input matrices  $(w_{ij})$  defined by

$$\nu_{\hat{\theta}, \theta; s}(dw) = \prod_{ij} w_{ij}^{-\hat{\theta}_i - \theta_j} \exp \left( - \sum_{i=1}^p \frac{s}{w_{i, p-i+1}} - \sum' \frac{1}{w_{ij}} \right) \prod_{ij} \frac{dw_{ij}}{w_{ij}},$$

where  $\hat{\theta}_i + \theta_j > 0$  for each  $i$  and  $j$ . Note that

$$\int_{(\mathbb{R}_{>0})^{n \times m}} \nu_{\hat{\theta}, \theta; s}(dw) = s^{-\sum_{i=1}^p (\hat{\theta}_i + \theta_{p-i+1})} \prod_{ij} \Gamma(\hat{\theta}_i + \theta_j).$$

Suppose  $W \in (\mathbb{R}_{>0})^{n \times m}$  and  $T = T(W) = (P, Q)$ . Define a mapping  $\sigma : (\mathbb{R}_{>0})^{n \times m} \rightarrow (\mathbb{R}_{>0})^p$  by

$$(3.11) \quad \sigma(W) = \text{sh } P = \text{sh } Q = (t_{nm}, t_{n-1, m-1}, \dots, t_{n-p+1, m-p+1}).$$

The next two corollaries are essentially a re-formulation of two of the main results in [11].

**COROLLARY 3.3.** *The push-forward of the measure  $\nu_{\hat{\theta}, \theta; s}$  under the geometric RSK map  $T$  is given by*

$$\nu_{\hat{\theta}, \theta; s} \circ T^{-1}(dt) = P^{-\theta} Q^{-\hat{\theta}} e^{-\mathcal{E}_s(T)} \prod_{ij} \frac{dt_{ij}}{t_{ij}}.$$

**PROOF.** This follows immediately from Theorems 3.1, 3.2 and the relation (3.5).  $\square$

**COROLLARY 3.4.** *The push-forward of  $\nu_{\hat{\theta}, \theta; s}$  under  $\sigma$  is given by*

$$\nu_{\hat{\theta}, \theta; s} \circ \sigma^{-1}(dx) = \begin{cases} \Psi_{\hat{\theta}}^p(x) \Psi_{\hat{\theta}; s}^p(x) \prod_{i=1}^p \frac{dx_i}{x_i} & n \geq m \\ \Psi_{\hat{\theta}; s}^p(x) \Psi_{\hat{\theta}}^p(x) \prod_{i=1}^p \frac{dx_i}{x_i} & n \leq m. \end{cases}$$

**PROOF.** This follows from Corollary 3.3 and the integral formula (2.5) for  $\Psi_{\lambda; s}^p$ .  $\square$

We also obtain from Theorems 3.1 and 3.2 the following integral identity. This is the analogue of the Cauchy-Littlewood identity in this setting.

**COROLLARY 3.5.** *Suppose  $s > 0$ ,  $\lambda \in \mathbb{C}^m$  and  $\nu \in \mathbb{C}^n$ , where  $n \geq m$  and  $\Re(\lambda_i + \nu_j) > 0$  for all  $i$  and  $j$ . Then*

$$(3.12) \quad \int_{(\mathbb{R}_{>0})^m} \Psi_{\nu; s}^m(x) \Psi_{\lambda}^m(x) \prod_{i=1}^m \frac{dx_i}{x_i} = s^{-\sum_{i=1}^m (\nu_i + \lambda_i)} \prod_{ij} \Gamma(\nu_i + \lambda_j).$$

PROOF. From the definitions (2.5), (3.8), the identity (3.5), Theorems 3.1 and 3.2, and Fubini's theorem,

$$\begin{aligned}
& s^{-\sum_{i=1}^m (\nu_i + \lambda_i)} \prod_{ij} \Gamma(\nu_i + \lambda_j) \\
&= \int_{(\mathbb{R}_{>0})^{n \times m}} \prod_{ij} w_{ij}^{-\nu_i - \lambda_j - 1} \exp \left( - \sum_{i=1}^m \frac{s}{w_{i,m-i+1}} - \sum' \frac{1}{w_{ij}} \right) \prod_{ij} dw_{ij} \\
&= \int_{(\mathbb{R}_{>0})^{n \times m}} P^{-\lambda} Q^{-\nu} e^{-\mathcal{E}_s(T)} \prod_{ij} \frac{dt_{ij}}{t_{ij}} \\
&= \int_{(\mathbb{R}_{>0})^m} \left( \int_{\Pi^n(x)} Q^{-\nu} e^{-\mathcal{F}_s(Q)} dQ \right) \left( \int_{\Pi^m(x)} P^{-\lambda} e^{-\mathcal{F}_0(P)} dP \right) \prod_{i=1}^m \frac{dx_i}{x_i} \\
&= \int_{(\mathbb{R}_{>0})^m} \Psi_{\nu;s}^m(x) \Psi_{\lambda}^m(x) \prod_{i=1}^m \frac{dx_i}{x_i},
\end{aligned}$$

as required.  $\square$

If  $n = m$  then (3.12) becomes:

COROLLARY 3.6. *Suppose  $s > 0$  and  $\lambda, \nu \in \mathbb{C}^n$ , where  $\Re(\lambda_i + \nu_j) > 0$  for all  $i$  and  $j$ . Then*

$$\int_{(\mathbb{R}_{>0})^n} e^{-s/x_n} \Psi_{\nu}^n(x) \Psi_{\lambda}^n(x) \prod_{i=1}^n \frac{dx_i}{x_i} = s^{-\sum_{i=1}^n (\nu_i + \lambda_i)} \prod_{ij} \Gamma(\nu_i + \lambda_j).$$

Using (3.13) this is equivalent to the following integral identity for  $GL(n, \mathbb{R})$ -Whittaker functions, due to Stade [24].

COROLLARY 3.7 (Stade). *Suppose  $r > 0$  and  $\lambda, \nu \in \mathbb{C}^n$ , where  $\Re(\lambda_i + \nu_j) > 0$  for all  $i$  and  $j$ . Then*

$$\int_{(\mathbb{R}_{>0})^n} e^{-rx_1} \Psi_{-\nu}^n(x) \Psi_{-\lambda}^n(x) \prod_{i=1}^n \frac{dx_i}{x_i} = r^{-\sum_{i=1}^n (\nu_i + \lambda_i)} \prod_{ij} \Gamma(\nu_i + \lambda_j).$$

We remark that, in fact, this identity is proved in [24] without assuming the condition  $\Re(\lambda_i + \nu_j) > 0$  for all  $i$  and  $j$ .

COROLLARY 3.8. *Suppose  $s > 0$  and  $\nu \in \mathbb{C}^n$  with  $\Re \nu_i > 0$  for each  $i$ . Then, for each  $m \leq n$ , the function  $\Psi_{\nu;s}^m$  is in  $L_2((\mathbb{R}_{>0})^m, \prod_{i=1}^m dx_i/x_i)$ , and the function  $e^{-sx_1} \Psi_{-\nu}^n(x)$  is in  $L_2((\mathbb{R}_{>0})^n, \prod_{i=1}^n dx_i/x_i)$ .*

PROOF. The first claim follows from Corollary 3.5 and the Plancherel theorem, as follows. We first note that, under the above hypotheses,

$$\hat{\Psi}_{\nu;s}^m(\lambda) = s^{-\sum_{i=1}^m (\nu_i + \lambda_i)} \prod_{ij} \Gamma(\nu_i + \lambda_j) \in L_2(\iota \mathbb{R}^m, s_m(\lambda) d\lambda).$$

This is easily verified using Stirling's approximation

$$\lim_{b \rightarrow \infty} |\Gamma(a + \iota b)| e^{\frac{\pi}{2}|b|} |b|^{\frac{1}{2}-a} = \sqrt{2\pi}, \quad a, b \in \mathbb{R}.$$

Now, suppose  $f \in L_2((\mathbb{R}_{>0})^m, \prod_{i=1}^m dx_i/x_i)$  such that  $\hat{f}$  is continuous and compactly supported on  $\iota\mathbb{R}^m$ . By the Plancherel theorem, such functions are dense in  $L_2((\mathbb{R}_{>0})^m, \prod_{i=1}^m dx_i/x_i)$  and, moreover, satisfy

$$(3.13) \quad f(x) = \int_{\iota\mathbb{R}^m} \hat{f}(\lambda) \Psi_\lambda^m(x) s_m(\lambda) d\lambda$$

almost everywhere. Indeed, for any  $g \in L_2((\mathbb{R}_{>0})^m, \prod_{i=1}^m dx_i/x_i)$  which is continuous and compactly supported we have, by Fubini's theorem,

$$\begin{aligned} \int_{(\mathbb{R}_{>0})^m} \left( \int_{\iota\mathbb{R}^m} \hat{f}(\lambda) \Psi_\lambda^m(x) s_m(\lambda) d\lambda \right) \overline{g(x)} \prod_{i=1}^m \frac{dx_i}{x_i} &= \int_{\iota\mathbb{R}^m} \hat{f}(\lambda) \overline{\hat{g}(\lambda)} s_m(\lambda) d\lambda \\ &= \int_{(\mathbb{R}_{>0})^m} f(x) \overline{g(x)} \prod_{i=1}^m \frac{dx_i}{x_i}. \end{aligned}$$

This implies (3.13). Now, by Corollary 3.5,

$$\int_{(\mathbb{R}_{>0})^m} |\Psi_{\nu;s}^m(x) \Psi_\lambda^m(x)| \prod_{i=1}^m \frac{dx_i}{x_i} \leq \int_{(\mathbb{R}_{>0})^m} \Psi_{\Re\nu;s}^m(x) \Psi_0^m(x) \prod_{i=1}^m \frac{dx_i}{x_i} < \infty.$$

It follows that, for  $f \in L_2((\mathbb{R}_{>0})^m, \prod_{i=1}^m dx_i/x_i)$  such that  $\hat{f}$  is continuous and compactly supported on  $\iota\mathbb{R}^m$ , the integral

$$\int_{(\mathbb{R}_{>0})^m} \int_{\iota\mathbb{R}^m} \Psi_{\nu;s}^m(x) \overline{\hat{f}(\lambda)} \Psi_\lambda^m(x) s_m(\lambda) d\lambda \frac{dx_i}{x_i}$$

is absolutely convergent, and so, by Fubini's theorem,

$$\begin{aligned} \int_{(\mathbb{R}_{>0})^m} \Psi_{\nu;s}^m(x) \overline{f(x)} \prod_{i=1}^m \frac{dx_i}{x_i} &= \int_{(\mathbb{R}_{>0})^m} \Psi_{\nu;s}^m(x) \left( \int_{\iota\mathbb{R}^m} \overline{\hat{f}(\lambda)} \Psi_\lambda^m(x) s_m(\lambda) d\lambda \right) \prod_{i=1}^m \frac{dx_i}{x_i} \\ &= \int_{\iota\mathbb{R}^m} \hat{\Psi}_{\nu;s}^m(\lambda) \overline{\hat{f}(\lambda)} s_m(\lambda) d\lambda. \end{aligned}$$

Hence, using the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \int_{(\mathbb{R}_{>0})^m} \Psi_{\nu;s}^m(x) \overline{f(x)} \prod_{i=1}^m \frac{dx_i}{x_i} \right| &= \left| \int_{\iota\mathbb{R}^m} \hat{\Psi}_{\nu;s}^m(\lambda) \overline{\hat{f}(\lambda)} s_m(\lambda) d\lambda \right| \\ &\leq \left( \int_{\iota\mathbb{R}^m} |\hat{\Psi}_{\nu;s}^m(\lambda)|^2 s_m(\lambda) d\lambda \right)^{1/2} \left( \int_{\iota\mathbb{R}^m} |\hat{f}(\lambda)|^2 s_m(\lambda) d\lambda \right)^{1/2}. \end{aligned}$$

This proves the first claim. The second claim follows from the first, letting  $m = n$  and using (3.13).  $\square$

Consider the probability measure on input matrices  $W$  defined by

$$\tilde{\nu}_{\hat{\theta},\theta;s}(dw) = Z_{\hat{\theta},\theta;s}^{-1} \nu_{\hat{\theta},\theta;s}(dw)$$

where

$$Z_{\hat{\theta}, \theta; s} = s^{-\sum_{i=1}^p (\hat{\theta}_i + \theta_i)} \prod_{ij} \Gamma(\hat{\theta}_i + \theta_j).$$

The following result was obtained in [11].

**COROLLARY 3.9.** *Suppose (w.l.o.g.) that  $n \geq m$ ,  $\theta_i < 0$  for each  $i$  and  $\hat{\theta}_j > 0$  for each  $j$ . Then, the Laplace transform of the law  $\tilde{\nu}_{\hat{\theta}, \theta; s} \circ t_{nm}^{-1}$  of the polymer partition function  $t_{nm}$  under  $\tilde{\nu}_{\hat{\theta}, \theta; s}$  is given by*

$$\int e^{-rt_{nm}} \tilde{\nu}_{\hat{\theta}, \theta; s}(dw) = \int_{\mathbb{R}^m} (rs)^{\sum_{i=1}^m (\theta_i - \lambda_i)} \prod_{ij} \Gamma(\lambda_i - \theta_j) \prod_{ij} \frac{\Gamma(\hat{\theta}_i + \lambda_j)}{\Gamma(\hat{\theta}_i + \theta_j)} s_n(\lambda) d\lambda.$$

**PROOF.** By Corollary 3.4,

$$\int e^{-rt_{nm}} \tilde{\nu}_{\hat{\theta}, \theta; s}(dw) = Z_{\hat{\theta}, \theta; s}^{-1} \int_{(\mathbb{R}_{>0})^m} e^{-rx_1} \Psi_{\theta}^m(x) \Psi_{\hat{\theta}; s}^m(x) \prod_{i=1}^m \frac{dx_i}{x_i}.$$

By Corollary 3.8, the functions  $e^{-rx_1} \Psi_{\theta}^m(x)$  and  $\Psi_{\hat{\theta}; s}^m(x)$  are in the space  $L_2((\mathbb{R}_{>0})^m, \prod_{i=1}^m dx_i/x_i)$ . The result follows, by Corollaries 3.6, 3.7 and the Plancherel theorem.  $\square$

#### 4. Equivalence of old and new description of geometric RSK

We explain here the equivalence of the Noumi-Yamada row insertion construction [19] and the definition of geometric RSK given in Section 3. The input weight matrix  $(w_{ij})$  is  $n \times m$ , where  $m$  is fixed and  $n$  represents time. After  $n$  time steps the Noumi-Yamada process gives two patterns  $P = \{z_{k\ell}\}$  and  $Q = \{z'_{ij}\}$ .  $P$  has height  $m$ ,  $Q$  has height  $n$ , and their common shape vector  $z_{m\cdot} = z'_{n\cdot}$  is of length  $p = m \wedge n$ . The rows of  $Q$  indexed by  $s = 1, \dots, n$  from top to bottom are the successive shape vectors (bottom rows)  $z_{m\cdot}(s) = (z_{m,\ell}(s))_{1 \leq \ell \leq m \wedge s}$  of the temporal evolution  $\{z(s) : 1 \leq s \leq n\}$  of the  $P$  pattern. Thus as in classic RSK the  $Q$  pattern serves as a recording pattern.

The Noumi-Yamada process begins with an empty pattern at time  $n = 0$ . Then the following step is repeated for  $n = 1, 2, 3, \dots$ .

*Noumi-Yamada construction for time step  $n - 1 \rightarrow n$ .* Let  $z = z(n - 1)$  denote the  $P$  pattern obtained after  $n - 1$  steps. Insertion of row  $w_{n\cdot}$  of weights into  $z$  transforms  $z$  into  $\tilde{z} = z(n)$  as follows.

(i) If  $n \geq m + 1$  (in other words, the triangle was filled by time  $n - 1$ ), then

$$(4.1) \quad \begin{aligned} a_{k,1} &= w_{n,k} && \text{for } 1 \leq k \leq m \\ a_{k+1,\ell+1} &= a_{k+1,\ell} \frac{z_{k+1,\ell} \tilde{z}_{k,\ell}}{\tilde{z}_{k+1,\ell} z_{k,\ell}} && \text{for } 1 \leq \ell \leq k < m \\ \tilde{z}_{k,\ell} &= a_{k,\ell} (z_{k,\ell} + \tilde{z}_{k-1,\ell}) && \text{for } 1 \leq \ell < k \leq m \\ \tilde{z}_{k,k} &= a_{k,k} z_{k,k} && \text{for } 1 \leq k \leq m. \end{aligned}$$

(ii) If  $n \leq m$ , then the equations above define  $\check{z}_{k,\ell}$  for  $1 \leq \ell \leq k \wedge (n-1)$ . Set

$$(4.2) \quad \check{z}_{k,n} = a_{n,n} \cdots a_{k,n} \quad \text{for } k = n, \dots, m,$$

while  $\check{z}_{k,\ell}$  for  $\ell \geq n+1$  remain undefined.

PROPOSITION 4.1. *Let  $(w_{ij})$  be an  $n \times m$  weight matrix and  $T = T(W)$  defined by (3.1). Then the output  $T$  is equivalent to the patterns  $(P, Q)$  obtained from  $n$  steps of the Noumi-Yamada evolution, through these equations:*

$$(4.3) \quad P \text{ pattern: } z_{k\ell} = t_{n-\ell+1, k-\ell+1}, \quad 1 \leq \ell \leq k \wedge n, 1 \leq k \leq m$$

$$(4.4) \quad Q \text{ pattern: } z'_{s\ell} = t_{s-\ell+1, m-\ell+1}, \quad 1 \leq \ell \leq m \wedge s, 1 \leq s \leq n.$$

Note in particular the common shape vector  $z_{m\bullet} = z'_{n\bullet} = (t_{n-\ell+1, m-\ell+1})_{1 \leq \ell \leq p}$ . Here is an illustration for  $n \times m = 3 \times 6$ .

$$(4.5) \quad T = \begin{bmatrix} z_{33} & z_{43} & z_{53} & z_{63} = z'_{33} & z'_{22} & z'_{11} \\ z_{22} & z_{32} & z_{42} & z_{52} & z_{62} = z'_{32} & z'_{21} \\ z_{11} & z_{21} & z_{31} & z_{41} & z_{51} & z_{61} = z'_{31} \end{bmatrix}$$

PROOF OF PROPOSITION 4.1. We keep  $m$  fixed and do induction on  $n$ . In the case  $n = 1$ , the  $m$ -vector  $\check{z}_{\bullet 1}$  described by equation (4.2) is the same as that obtained by applying  $R_1 = \pi_1^m = l_{1m} \circ \cdots \circ l_{11}$  to the top row  $w_{1\bullet}$  of the weight matrix.

Suppose the statement is true for  $T^{n-1, m}$ . Add the  $n$ th weight row  $w_{n\bullet}$  to  $T^{n-1, m}$  and call the resulting  $n \times m$  matrix  $\tilde{T}^{n, m} = \begin{pmatrix} T^{n-1, m} \\ w_{n\bullet} \end{pmatrix}$ . Then  $T^{n, m} = R_n(\tilde{T}^{n, m})$ . From the definition of  $R_n$  we see that on row  $i \in \{1, \dots, n-1\}$  it alters only elements  $\tilde{t}_{ij}$  for  $j-i \leq m-n$ . Consequently after the application of  $R_n$ , the induction assumption implies that (4.4) remains in force for  $1 \leq s \leq n-1$ . It only remains to check that (4.3) holds after the application of  $R_n$ .

Again we do induction, starting from the bottom row of  $T^{n, m}$  and moving up row by row. This corresponds to executing  $R_n = \pi_{(n-m) \vee 0+1}^{(m-n) \vee 0+1} \circ \cdots \circ \pi_{n-1}^{m-1} \circ \pi_n^m$  step by step.

Before applying  $\pi_n^m$ , the two bottom rows of  $\tilde{T}^{n, m}$  are

$$\tilde{T}^{n, m} = \begin{bmatrix} \cdots & \cdots & & \\ z_{11} & z_{21} & \cdots & z_{m1} \\ w_{n1} & w_{n2} & \cdots & w_{nm} \end{bmatrix} = \begin{bmatrix} \cdots & \cdots & & \\ z_{11} & z_{21} & \cdots & z_{m1} \\ a_{11} & a_{21} & \cdots & a_{m1} \end{bmatrix}$$



where we used the first row of (4.1). Apply  $\pi_n^m = l_{nm} \circ l_{n,m-1} \circ \cdots \circ l_{n1}$ . Only the bottom two rows are impacted. Use the notation from (4.1).

$$\begin{array}{ccc}
\begin{bmatrix} z_{11} & z_{21} & z_{31} & \cdots & z_{m1} \\ a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \end{bmatrix} & \xrightarrow{l_{n1}} & \begin{bmatrix} z_{11} & z_{21} & z_{31} & \cdots & z_{m1} \\ \check{z}_{11} & a_{21} & a_{31} & \cdots & a_{m1} \end{bmatrix} \\
\downarrow l_{n2} & & \downarrow l_{n3} \\
\begin{bmatrix} a_{22} & z_{21} & z_{31} & \cdots & z_{m1} \\ \check{z}_{11} & \check{z}_{21} & a_{31} & \cdots & a_{m1} \end{bmatrix} & & \begin{bmatrix} a_{22} & a_{32} & z_{31} & \cdots & z_{m1} \\ \check{z}_{11} & \check{z}_{21} & \check{z}_{31} & \cdots & a_{m1} \end{bmatrix} \\
\downarrow l_{n4} & \cdots & \downarrow l_{nm} \\
& & \begin{bmatrix} a_{22} & a_{32} & a_{42} & \cdots & a_{m2} & z_{m1} \\ \check{z}_{11} & \check{z}_{21} & \check{z}_{31} & \cdots & \check{z}_{m-1,1} & \check{z}_{m1} \end{bmatrix}
\end{array}$$

Now the bottom row of  $T^{n,m}$  is in place. Note that the transformations above left in place  $z_{m1} = z'_{n1}$  as they should, for this entry is in accordance with (4.4).

Next, an application of  $\pi_{n-1}^{m-1} = l_{n-1,m-1} \circ l_{n-1,m-2} \circ \cdots \circ l_{n-1,1}$  transforms rows  $n-2$  and  $n-1$  in this manner:

$$\begin{array}{ccc}
\begin{bmatrix} z_{22} & z_{32} & z_{42} & \cdots & z_{m-1,2} & z'_{n2} & z'_{n-1,1} \\ a_{22} & a_{32} & a_{42} & \cdots & a_{m-1,2} & a_{m2} & z'_{n1} \\ \check{z}_{11} & \check{z}_{21} & \check{z}_{31} & \cdots & \check{z}_{m-2,1} & \check{z}_{m-1,1} & \check{z}_{m1} \end{bmatrix} \\
\downarrow \pi_{n-1}^{m-1} \\
\begin{bmatrix} a_{33} & a_{43} & a_{53} & \cdots & a_{m-2,3} & z'_{n2} & z'_{n-1,1} \\ \check{z}_{22} & \check{z}_{32} & \check{z}_{42} & \cdots & \check{z}_{m-1,2} & \check{z}_{m2} & z'_{n1} \\ \check{z}_{11} & \check{z}_{21} & \check{z}_{31} & \cdots & \check{z}_{m-2,1} & \check{z}_{m-1,1} & \check{z}_{m1} \end{bmatrix}
\end{array}$$

The bottom two rows of  $T^{n,m}$  are in place. These steps continue until we arrive at  $T^{n,m}$ .  $\square$

## 5. Symmetric input matrix

As it is needed in the following, we will write  $R_i^{n,m}$  and  $T = T^{n,m}$  for the mappings defined in the previous section, and note the following recursive structure. Let  $W = (w_{ij}) \in (\mathbb{R}_{>0})^{n \times m}$  and write  $W_{k,m} = (w_{ij}, 1 \leq i \leq k, 1 \leq j \leq m)$ . Recall that

$$T^{n,m} = R_n^{n,m} \circ R_{n-1}^{n,m} \circ \cdots \circ R_1^{n,m}.$$

Now, for each  $i \leq n$ , the mapping  $R_i^{n,m}$  acts only on the first  $i$  rows of  $W$  and leaves the remaining rows of  $W$  unchanged. In fact, for each  $i \leq k \leq n$ , we have

$$R_i^{n,m}(W) = \begin{pmatrix} R_i^{k,m}(W_{k,m}) \\ W_{k,m}^c \end{pmatrix},$$

where  $W_{k,m}^c = (w_{ij}, k+1 \leq i \leq n, 1 \leq j \leq m)$ . This property is immediate from the definitions. This gives the basic recursion

$$(5.1) \quad T^{n,m}(W) = R_n^{n,m} \begin{pmatrix} T^{n-1,m}(W_{n-1,m}) \\ w_{n1} \quad \cdots \quad w_{nm} \end{pmatrix}.$$

Recall that

$$(5.2) \quad T^{m,n}(W^t) = [T^{n,m}(W)]^t.$$

In particular, if  $n = m$  and  $W$  is symmetric, then  $T^{n,n}(W)$  is also symmetric.

LEMMA 5.1. *Suppose that  $n = m$  and  $W$  is symmetric.*

(a) *The following recursion holds:*

$$(5.3) \quad T^{n,n}(W) = R_n^{n,n} \left( \begin{bmatrix} R_n^{n,n-1} \left( T^{n-1,n-1}(W_{n-1,n-1}) \right) \\ w_{1n} \dots w_{n-1,n} \\ w_{1n} \dots w_{nn} \end{bmatrix} \right)^t.$$

Moreover, if we denote by  $(s_{ij})$  the elements of the  $(n-1) \times n$  matrix

$$(5.4) \quad S = \begin{bmatrix} R_n^{n,n-1} \left( T^{n-1,n-1}(W_{n-1,n-1}) \right) \\ w_{1n} \dots w_{n-1,n} \end{bmatrix}^t$$

and by  $(t_{ij})$  the elements of  $T^{n,n}(W)$ , then

$$(5.5) \quad \begin{aligned} t_{ij} &= s_{ij} && \text{for } 1 \leq i < j \leq n \\ t_{11} &= s_{12}/2s_{11} \\ t_{ii} &= s_{i,i+1}s_{i-1,i}/s_{ii} && \text{for } 2 \leq i \leq n-1 \\ t_{nn} &= 2s_{n-1,n}w_{nn}. \end{aligned}$$

(b) *For  $n \geq 1$  we have this identity:*

$$(5.6) \quad 4^{\lfloor n/2 \rfloor} \prod_{i=1}^n w_{ii} = \frac{\prod_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} t_{n-2j, n-2j}}{\prod_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} t_{n-1-2j, n-1-2j}} = \frac{\prod_{i \text{ odd}} z_{ni}}{\prod_{i \text{ even}} z_{ni}}.$$

PROOF. Part (a). Using (5.1), (5.2) and the fact the  $W$  is symmetric,

$$\begin{aligned} T^{n,n}(W) &= R_n^{n,n} \begin{pmatrix} T^{n-1,n}(W_{n-1,n}) \\ w_{n1} \dots w_{nn} \end{pmatrix} \\ &= R_n^{n,n} \begin{pmatrix} [T^{n,n-1}([W_{n-1,n}]^t)]^t \\ w_{n1} \dots w_{nn} \end{pmatrix} \\ &= R_n^{n,n} \begin{pmatrix} [T^{n,n-1}(W_{n,n-1})]^t \\ w_{n1} \dots w_{nn} \end{pmatrix} \\ &= R_n^{n,n} \left( \begin{bmatrix} R_n^{n,n-1} \left( T^{n-1,n-1}(W_{n-1,n-1}) \right) \\ w_{n1} \dots w_{n,n-1} \end{bmatrix} \right)^t \\ &= R_n^{n,n} \left( \begin{bmatrix} R_n^{n,n-1} \left( T^{n-1,n-1}(W_{n-1,n-1}) \right) \\ w_{1n} \dots w_{n-1,n} \\ w_{1n} \dots w_{nn} \end{bmatrix} \right)^t. \end{aligned}$$

This proves the first claim. So we have

$$T^{n,n}(W) = R_n^{n,n} \begin{pmatrix} S \\ w_{1n} \dots w_{nn} \end{pmatrix},$$

where  $S \in (\mathbb{R}_{>0})^{(n-1) \times n}$ . To prove the second claim, first note that the mapping  $R_n^{n,n}$  leaves the elements of its input matrix which are strictly above the diagonal unchanged. Thus,  $t_{ij} = s_{ij}$  for  $1 \leq i < j \leq n$ . Using this, the symmetry of  $T$ , and recalling how the row insertion procedure works (see Section 4), we see that

$$t_{nn} = w_{nn}(t_{n-1,n} + s_{n-1,n}) = 2s_{n-1,n}w_{nn},$$

$$\begin{aligned} t_{n-1,n-1} &= \frac{t_{n-1,n}s_{n-1,n}}{s_{n-1,n-1}(t_{n-1,n} + s_{n-1,n})}(t_{n-2,n-1} + s_{n-2,n-1}) \\ &= s_{n-1,n}s_{n-2,n-1}/s_{n-1,n-1}, \end{aligned}$$

and so on; for  $2 \leq i \leq n-1$  we have  $t_{ii} = s_{i,i+1}s_{i-1,i}/s_{ii}$  and then finally,

$$t_{11} = \frac{t_{12}s_{12}}{s_{11}(t_{12} + s_{12})} = s_{12}/2s_{11},$$

as required.

Part (b). The second equality in (5.6) is a consequence of (4.3). The first equality is proved by induction on  $n$ . Cases  $n = 2$  and  $n = 3$  are checked by hand.

Suppose (5.6) is true for  $n-1$ . Observe first from the definition of the mappings that  $R_n^{n,n-1}$  operating on  $\begin{pmatrix} T^{n-1,n-1} \\ w_{1n} \dots w_{n-1,n} \end{pmatrix}$  does not alter the diagonal  $\{t_{ii}^{n-1}\}_{1 \leq i \leq n-1}$  of  $T^{n-1,n-1}$ . Consequently (5.4) implies that  $s_{ii} = t_{ii}^{n-1}$  for  $1 \leq i \leq n-1$ .

Suppose  $n$  is even. Then the middle fraction of (5.6) develops as follows, through equations (5.5),  $s_{ii} = t_{ii}^{n-1}$  and by induction:

$$\begin{aligned} \frac{t_{nn}t_{n-2,n-2} \cdots t_{22}}{t_{n-1,n-1}t_{n-3,n-3} \cdots t_{11}} &= \frac{2s_{n-1,n}w_{nn} \cdot \frac{s_{n-2,n-1}s_{n-3,n-2} \cdots s_{23}s_{12}}{s_{n-2,n-2}} \cdot \frac{s_{22}}{s_{11}}}{\frac{s_{n-1,n}s_{n-2,n-1}}{s_{n-1,n-1}} \cdot \frac{s_{n-3,n-2}s_{n-4,n-3} \cdots s_{12}}{s_{n-3,n-3}} \cdots \frac{s_{12}}{2s_{11}}} \\ &= 4w_{nn} \cdot \frac{s_{n-1,n-1}s_{n-3,n-3} \cdots s_{11}}{s_{n-2,n-2}s_{n-4,n-4} \cdots s_{22}} \\ &= 4w_{nn} \cdot 4^{\frac{n}{2}-1} \prod_{i=1}^{n-1} w_{ii} = 4^{\lfloor n/2 \rfloor} \prod_{i=1}^n w_{ii}. \end{aligned}$$

The case of odd  $n$  develops similarly except that now the product in the numerator finishes with  $s_{12}/2s_{11}$  and consequently the factors of 2 cancel each other.  $\square$

**THEOREM 5.2.** *Suppose that  $n = m$  and  $W$  is symmetric. Then  $T = T(W) = (t_{ij})$  is also symmetric, and the Jacobian of the map*

$$(\log w_{ij}, 1 \leq i \leq j \leq n) \mapsto (\log t_{ij}, 1 \leq i \leq j \leq n)$$

*is  $\pm 1$ .*

**PROOF.** We prove this by induction on  $n$ . When  $n = 2$ , we have  $t_{11} = w_{12}/2$ ,  $t_{12} = w_{11}w_{12}$ ,  $t_{22} = 2w_{11}w_{12}w_{22}$  and the result is immediate. Now, by the previous lemma,

$$T = R_n^{n,n} \left( \left[ R_n^{n,n-1} \begin{pmatrix} T^{n-1,n-1}(W_{n-1,n-1}) \\ w_{1n} \dots w_{n-1,n} \end{pmatrix} \right]^t \right).$$

Denoting by  $(s_{ij})$  the elements of the matrix

$$S = \left[ R_n^{n,n-1} \begin{pmatrix} T^{n-1,n-1}(W_{n-1,n-1}) \\ w_{1n} \dots w_{n-1,n} \end{pmatrix} \right]^t,$$

we have, by the previous lemma,

$$(5.7) \quad \begin{aligned} t_{ij} &= s_{ij} && \text{for } 1 \leq i < j \leq n \\ t_{11} &= s_{12}/2s_{11} \\ t_{ii} &= s_{i,i+1}s_{i-1,i}/s_{ii} && \text{for } 2 \leq i \leq n-1 \\ t_{nn} &= 2s_{n-1,n}w_{nn} \end{aligned}$$

This expresses the  $n(n+1)/2$  variables  $t_{ij}$ ,  $1 \leq i \leq j \leq n$  as a function, which we shall denote by  $F$ , of the  $n(n+1)/2$  variables  $s_{ij}$ ,  $1 \leq i < j \leq n$  and  $s_{11}, \dots, s_{n-1,n-1}, w_{nn}$ .

Denote by  $t_{ij}^{n-1}$  the elements of the symmetric matrix  $T^{n-1,n-1}(W_{n-1,n-1})$ . By the induction hypothesis, the map

$$(\log w_{ij}, 1 \leq i \leq j \leq n-1) \mapsto (\log t_{ij}^{n-1}, 1 \leq i \leq j \leq n-1)$$

has Jacobian  $\pm 1$ . The mapping  $R_n^{n,n-1}$  on the whole of  $(\mathbb{R}_{>0})^{n \times (n-1)}$  is a composition of  $l_{ij}$ -maps and hence has Jacobian  $\pm 1$  in logarithmic variables; since it leaves matrix elements above the diagonal unchanged, its restriction to the space of matrix elements on and below the diagonal also has Jacobian  $\pm 1$  in logarithmic variables. It follows that the mapping

$$\begin{aligned} (\log w_{ij}, 1 \leq i \leq j < n; \log w_{in}, 1 \leq i < n) \\ \mapsto (\log s_{ij}, 1 \leq i < j \leq n; \log s_{ii}, 1 \leq i < n) \end{aligned}$$

has Jacobian  $\pm 1$ . It therefore remains only to show that the Jacobian submatrix of the map  $F$  (in logarithmic variables) which corresponds to the

variables  $(\log s_{11}, \dots, \log s_{n-1,n-1}, \log w_{nn})$  and  $(\log t_{11}, \dots, \log t_{nn})$  has determinant  $\pm 1$ . From (5.7), this sub matrix is given by

$$\begin{matrix} & \log s_{11} & \log s_{22} & \cdots & \log s_{n-1,n-1} & \log w_{nn} \\ \log t_{11} & & -1 & & & \\ \log t_{22} & & & -1 & & \\ \vdots & & & & \ddots & \\ \log t_{n-1,n-1} & & & & & -1 \\ \log t_{n,n} & & & & & & 1 \end{matrix},$$

which completes the proof.  $\square$

Consider the measure on symmetric input matrices with positive entries defined by

$$\nu_{\alpha;\delta}(dw) = \prod_{i < j} w_{ij}^{-\alpha_i - \alpha_j} \prod_i w_{ii}^{-\alpha_i - \delta} \exp \left( - \sum_{i < j} \frac{1}{w_{ij}} - \sum_i \frac{1}{2w_{ii}} \right) \prod_{i \leq j} \frac{dw_{ij}}{w_{ij}},$$

where  $\alpha \in \mathbb{R}^n$  and  $\delta \in \mathbb{R}$  satisfy  $\alpha_i + \delta > 0$  for each  $i$  and  $\alpha_i + \alpha_j > 0$  for  $i \neq j$ . Note that

$$\int_{(\mathbb{R}_{>0})^{n(n+1)/2}} \nu_{\alpha;\delta}(dw) = 2^{\sum_{i=1}^n (\alpha_i + \delta)} \prod_i \Gamma(\alpha_i + \delta) \prod_{i < j} \Gamma(\alpha_i + \alpha_j).$$

In this setting we have  $R = C$  and so, using (3.5) and Lemma 5.1(b),

$$\prod_{i < j} w_{ij}^{-\alpha_i - \alpha_j} \prod_i w_{ii}^{-\alpha_i - \delta} = 4^{\lfloor n/2 \rfloor \delta} \prod_i z_{ni}^{(-1)^i \delta} R^{-\alpha}.$$

Thus, by Theorems 3.2 and 5.2, we obtain the following result.

**COROLLARY 5.3.** *The push-forward of  $\nu_{\alpha;\delta}$  under  $\sigma$  is given by*

$$\nu_{\alpha;\delta} \circ \sigma^{-1}(dx) = 4^{\lfloor n/2 \rfloor \delta} f(x)^\delta e^{-1/2x_n} \Psi_\alpha^n(x) \prod_{i=1}^n \frac{dx_i}{x_i},$$

where

$$f(x) = \prod_i x_i^{(-1)^i}.$$

Furthermore, if  $\lambda \in \mathbb{C}^n$  and  $\gamma \in \mathbb{C}$  satisfy  $\Re(\lambda_i + \gamma) > 0$  for each  $i$ , and  $\Re(\lambda_i + \lambda_j) > 0$  for  $i \neq j$ , then

$$\begin{aligned} \int_{(\mathbb{R}_{>0})^n} f(x)^\gamma e^{-1/2x_n} \Psi_\lambda^n(x) \prod_{i=1}^n \frac{dx_i}{x_i} \\ = 4^{-\lfloor n/2 \rfloor \gamma} 2^{\sum_{i=1}^n (\lambda_i + \gamma)} \prod_i \Gamma(\lambda_i + \gamma) \prod_{i < j} \Gamma(\lambda_i + \lambda_j). \end{aligned}$$

Now, using (2.7) we can strengthen this to:

COROLLARY 5.4. *Suppose  $\lambda \in \mathbb{C}^n$  and  $\gamma \in \mathbb{C}$  satisfy  $\Re(\lambda_i + \gamma) > 0$  for each  $i$ , and  $\Re(\lambda_i + \lambda_j) > 0$  for  $i \neq j$ . Then, for  $s > 0$ ,*

$$\begin{aligned} \int_{(\mathbb{R}_{>0})^n} f(x)^\gamma e^{-s/x_n} \Psi_\lambda^n(x) \prod_{i=1}^n \frac{dx_i}{x_i} \\ = c_n(s, \gamma) s^{-\sum_{i=1}^n \lambda_i} \prod_i \Gamma(\lambda_i + \gamma) \prod_{i < j} \Gamma(\lambda_i + \lambda_j), \end{aligned}$$

where

$$c_n(s, \gamma) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ s^{-\gamma} & \text{if } n \text{ is odd.} \end{cases}$$

By (3.13) this is equivalent to the following identity.

COROLLARY 5.5. *Suppose  $\lambda \in \mathbb{C}^n$  and  $\gamma \in \mathbb{C}$  satisfy  $\Re(\lambda_i + \gamma) > 0$  for each  $i$ , and  $\Re(\lambda_i + \lambda_j) > 0$  for  $i \neq j$ . Then, for  $s > 0$ ,*

$$\begin{aligned} \int_{(\mathbb{R}_{>0})^n} f(x')^\gamma e^{-sx_1} \Psi_{-\lambda}^n(x) \prod_{i=1}^n \frac{dx_i}{x_i} \\ = c_n(s, \gamma) s^{-\sum_{i=1}^n \lambda_i} \prod_i \Gamma(\lambda_i + \gamma) \prod_{i < j} \Gamma(\lambda_i + \lambda_j), \end{aligned}$$

where  $x'_i = 1/x_{n-i+1}$ .

Note that  $f(x') = f(x)$  if  $n$  is even and  $f(x') = 1/f(x)$  if  $n$  is odd.

Now, consider the probability measure on symmetric matrices with positive entries defined by

$$\tilde{\nu}_{\alpha; \delta}(dw) = Z_{\alpha; \delta}^{-1} \nu_{\alpha; \delta}(dw),$$

where

$$Z_{\alpha; \delta} = 2^{\sum_{i=1}^n (\alpha_i + \delta)} \prod_i \Gamma(\alpha_i + \delta) \prod_{i < j} \Gamma(\alpha_i + \alpha_j).$$

From Corollary 5.3, we obtain:

COROLLARY 5.6. *The Laplace transform of the law of the polymer partition function  $t_{nn}$  under  $\tilde{\nu}_{\alpha; \delta}$  is given for  $r > 0$  by*

$$\int e^{-rt_{nn}} \tilde{\nu}_{\alpha; \delta}(dw) = 4^{\lfloor n/2 \rfloor \delta} Z_{\alpha; \delta}^{-1} \int_{(\mathbb{R}_{>0})^n} f(x)^\delta e^{-rx_1 - 1/2x_n} \Psi_\alpha^n(x) \prod_{i=1}^n \frac{dx_i}{x_i}.$$

It follows from Corollary 3.6 (or 3.8) that the function  $e^{-1/2x_n} \Psi_\alpha^n(x)$  is in  $L_2((\mathbb{R}_{>0})^n, \prod_{i=1}^n dx_i/x_i)$ . Moreover, by Corollary 3.6, for  $\lambda \in \mathbb{C}^n$ ,

$$\int_{(\mathbb{R}_{>0})^n} e^{-1/2x_n} \Psi_\alpha^n(x) \Psi_\lambda^n(x) \prod_{i=1}^n \frac{dx_i}{x_i} = 2^{\sum_i (\lambda_i + \alpha_i)} \prod_{i,j} \Gamma(\alpha_i + \lambda_j).$$

Thus, by the Plancherel theorem, for any  $g \in L_2((\mathbb{R}_{>0})^n, \prod_{i=1}^n dx_i/x_i)$  we can write

$$\begin{aligned} \int_{(\mathbb{R}_{>0})^n} g(x) e^{-1/2x_n} \Psi_\alpha^n(x) \prod_{i=1}^n \frac{dx_i}{x_i} \\ = \int_{\mathbb{C}^n} \overline{\hat{g}(\lambda)} 2^{\sum_i (\lambda_i + \alpha_i)} \prod_{i,j} \Gamma(\alpha_i + \lambda_j) s_n(\lambda) d\lambda. \end{aligned}$$

For well-behaved functions  $g$  (for example if  $\hat{g}(\lambda)$  is analytic and decays rapidly as  $|\lambda| \rightarrow \infty$ , uniformly in a vertical strip containing the origin) the vertical contours of integration can be shifted to the right, so that  $\Re \lambda_i > 0$  for each  $i$ .

Suppose  $n$  is even. By Corollary 5.5, if  $r > 0$  and  $\Re \lambda_i > 0$  for each  $i$ ,

$$\begin{aligned} \int_{(\mathbb{R}_{>0})^n} f(x)^\delta e^{-rx_1} \Psi_{-\lambda}^n(x) \prod_{i=1}^n \frac{dx_i}{x_i} \\ = r^{-\sum_{i=1}^n \lambda_i} \prod_i \Gamma(\lambda_i + \delta) \prod_{i < j} \Gamma(\lambda_i + \lambda_j). \end{aligned}$$

Formally, this yields the following integral formula for the Laplace transform of the law of the polymer partition function  $t_{nn}$  under the probability measure  $\tilde{\nu}_{\alpha;\delta}$ :

$$\begin{aligned} (5.8) \quad \int e^{-rt_{nn}} \tilde{\nu}_{\alpha;\delta}(dw) \\ = \int \left(\frac{r}{2}\right)^{-\sum_i \lambda_i} \prod_i \frac{\Gamma(\lambda_i + \delta)}{\Gamma(\alpha_i + \delta)} \prod_{i,j} \Gamma(\alpha_i + \lambda_j) \prod_{i < j} \frac{\Gamma(\lambda_i + \lambda_j)}{\Gamma(\alpha_i + \alpha_j)} s_n(\lambda) d\lambda \end{aligned}$$

where the contours of integration are along vertical lines with  $\Re \lambda_i > 0$  for each  $i$ . It seems reasonable to expect this integral formula to be valid, at least in some suitably regularised sense.

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